

The complete problem:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

B.C. $\begin{cases} T|_{x=0} = T_0 \\ T|_{x=a} = \text{finite} \end{cases}$

$$\begin{cases} -k \frac{\partial T}{\partial y}|_{y=b} = h(T|_{y=b} - T_\infty) \\ k \frac{\partial T}{\partial y}|_{y=-b} = h(T|_{y=-b} - T_\infty) \end{cases}$$

define: $\theta(x, y) = T(x, y) - T_\infty$.

therefore: $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$

B.C. $\begin{cases} \theta|_{x=0} = T_0 - T_\infty = \theta_0 \\ \theta|_{x=a} = \text{finite} \end{cases}$

← Nonhomogeneous.

$$\begin{cases} -k \frac{\partial \theta}{\partial y}|_{y=b} = h \theta|_{y=b} \\ k \frac{\partial \theta}{\partial y}|_{y=-b} = h \theta|_{y=-b} \end{cases}$$

①. Assume: $\theta(x, y) = X(x)Y(y) \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0$

then: $\begin{cases} X'' - \mu X = 0 \\ Y'' + \mu Y = 0 \end{cases}$

② Solving ODEs

(1) Case 1: $\mu < 0 \Rightarrow \mu = -\lambda^2 \quad (\lambda > 0)$

$$\begin{cases} X'' + \lambda^2 X = 0 \\ Y'' - \lambda^2 Y = 0 \end{cases} \Rightarrow \begin{cases} X(x) = A \cos \lambda x + B \sin \lambda x \\ Y(y) = C \cosh \lambda y + D \sinh \lambda y \end{cases}$$

Imposing B.C. From symmetry of the problem: $\theta(-y) = \theta(y)$

so: $Y(y) = C \cosh \lambda y$ as $D = 0$

Imposing B.C. $-k \frac{\partial \theta}{\partial y} \Big|_{y=b} = h \theta \Big|_{y=b} \Rightarrow -k \frac{dY}{dy} \Big|_{y=b} = h Y \Big|_{y=b}$

so: $-k C \lambda \sinh \lambda b = h C \cosh \lambda b$

i.e.: $C (k \lambda \sinh \lambda b + h \cosh \lambda b) = 0$

Therefore: $C = 0$ not a meaningful solution!

Conclusion: μ cannot be < 0 !

(2) Case 2: $\mu = 0 \Rightarrow$

$$\begin{cases} X'' = 0 \\ Y'' = 0 \end{cases} \Rightarrow \begin{cases} X(x) = Ax + B \\ Y(y) = Cy + D \end{cases}$$

Imposing B.C. From symmetry: $\theta(-y) = \theta(y)$

so: $Y(y) = D$ as $C = 0$

Imposing B.C. $-k \frac{\partial \theta}{\partial y} \Big|_{y=b} = h \theta \Big|_{y=b} \Rightarrow -k \frac{dY}{dy} \Big|_{y=b} = h Y \Big|_{y=b}$

so: $0 = h D$

Therefore: $D = 0$ not a meaningful solution!

Conclusion: $\mu \neq 0$!

(3) Case 3. $\mu > 0 \Rightarrow \mu = \lambda^2$ ($\lambda > 0$)

$$\begin{cases} X'' - \lambda^2 X = 0 \\ Y'' + \lambda^2 Y = 0 \end{cases} \Rightarrow \begin{cases} X(x) = Ae^{\lambda x} + Be^{-\lambda x} \\ Y(y) = C \cos \lambda y + D \sin \lambda y \end{cases}$$

Imposing B.C.: From symmetry: $\theta(-y) = \theta(y)$

so: $Y(y) = C \cos \lambda y$ as $D=0$

Imposing B.C. $\theta|_{x=0} = \text{finite} \Rightarrow X|_{x=0} = \text{finite}$

so: $X(x) = Be^{-\lambda x}$ as $A=0$ ($e^{\lambda x} \rightarrow \infty$)

Imposing B.C.: $-k \frac{\partial \theta}{\partial y} \Big|_{y=b} = h \theta \Big|_{y=b} \Rightarrow -k \frac{dY}{dy} \Big|_{y=b} = h Y \Big|_{y=b}$

so: $-k \cdot (-\lambda C \sin \lambda b) = h C \cos \lambda b$

i.e.: $C(\lambda k \sin \lambda b - h \cos \lambda b) = 0$

C cannot be 0, otherwise $Y(y) = 0$ not meaningful solⁿ.

Therefore: $\lambda k \sin \lambda b - h \cos \lambda b = 0$

i.e.: $\boxed{\cot(\lambda b) = \frac{\lambda k}{h}}$

λ can only take certain values (eigenvalues)

$\lambda = \lambda_n, n=1, 2, 3, \dots$

For each n : $\theta_n(x, y) = C_n \cos \lambda_n y e^{-\lambda_n x}$

③ Making final solution:

$$\theta(x,y) = \sum_{n=1}^{\infty} C_n \cos \lambda_n y e^{-\lambda_n x}$$

with λ_n defined by: $\cot \lambda b = \frac{\lambda k}{h}$

④ Determining the unknown coefficient:

Applying Nonhomogeneous B.C., $\theta|_{x=0} = \theta_0$

$$\text{So, } \theta_0 = \sum_{n=1}^{\infty} C_n \cos \lambda_n y$$

Multiplying each side by $\cos \lambda_m y$ and integrate from 0 to b:

$$\int_0^b \theta_0 \cos \lambda_m y dy = \int_0^b \sum_{n=1}^{\infty} C_n \cos \lambda_n y \cos \lambda_m y dy$$

i.e., $\theta_0 \int_0^b \cos \lambda_m y dy = C_m \int_0^b \cos^2 \lambda_m y dy$ after using the orthogonal property

$$C_n = \frac{\theta_0 \int_0^b \cos \lambda_n y dy}{\int_0^b \cos^2 \lambda_n y dy} = \frac{\theta_0 \sin \lambda_n b}{\lambda_n \left(\frac{b}{2} + \frac{\sin 2\lambda_n b}{4\lambda_n} \right)}$$

Therefore: $T(x,y) = T_{\infty} + (T_0 - T_{\infty}) \sum_{n=1}^{\infty} \frac{\sin \lambda_n b}{\lambda_n b + \frac{\sin 2\lambda_n b}{4}} \cos \lambda_n y e^{-\lambda_n x}$